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Wave Equation for a Magnetic Monopole

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We show that there is room, in the Dirac equation, for a massless monopole. The basic idea is that the Dirac equation admits a second electromagnetic minimal coupling associated to the chiral gauge $e^{i\gamma_5\theta}$, which is only valid for a massless particle, but satisfies all the symmetry laws of a monopole. In the problem of the diffusion on a central electric field, we find the Poincaré integral and the Dirac relation $eg/\hbar c = n/2$. The latter is deduced as a consequence of the fact (which is shown in this paper) that eg/c is the projection of the total angular momentum on the symmetry axis of the system formed by the monopole and the electric charge. Another important property is that a monopole and an antimonopole have opposite helicities (as for the neutrino), but do not have opposite charges: this precludes a vacuum magnetic polarization which would be analogous to the electric one, but allows us to imagine an aether made up of monopole-antimonopole pairs. The theory is then generalized on the basis of a nonlinear equation which is the most general invariant equation under the chiral gauge law. This equation admits solutions corresponding to massive monopoles, among which there are bradyons (i.e., ordinary massive particles) and tachyons. This equation is shown to be closely related to previous works initiated by Hermann Weyl, on Dirac's theory in the framework of general relativity. In conclusion, it is suggested that massless monopoles are perhaps excited states of the neutrino and that they may be produced in some weak interactions. Consequences on the solar activity are considered.

1. INTRODUCTION

During the last ten years, most of the works devoted to Dirac's hypothesis on the possible existence of magnetic monopoles were directed toward two trends. (1) A heavy monopole initially suggested by 't Hooft (G. 't Hooft, 1974) and Polyakov (Polyakov, 1974) in the framework of $SU(2)$ gauge theories (see Burzlaff, 1983; Jaffe and Taubes, 1980). (2) The theory of the motion of an electric charge in the central field of a fixed monopole and the elimination of the Dirac strings using a procedure initiated by Wu and Yang on the basis of the theory of fiberbundles (Wu and Yang, 1975; Kazama, Yang, and Goldhaber, 1977; Yamagishi, 1983).

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The problems considered in the present paper are quite different. Our aim is to find a spinorial wave equation for a magnetic monopole in the framework of the Dirac equation. This implies that we find the corresponding electromagnetic coupling or, equivalently, to find a new local gauge law different from the common phase invariance. We will show that the Dirac equation admits two and only two possible invariant gauges. The first one is the phase invariance $e^{i\theta}$ which gives the electromagnetic coupling with an electric charge. The second one is the chiral transformation $e^{i\gamma_5\theta}$, only valid for a massless particle, which is known in the neutrino theory (Touschek, 1957; Pauli, 1957), but was previously only used in its global form ($\theta = \text{const.}$). It will be shown that its extension to a local gauge law provides us with a new electromagnetic coupling which corresponds to a *massless magnetic monopole.*

The fact that the particle so described truly represents a monopole will be proved by its motion in a central electric field, which has all the right properties, including the Dirac condition $(eg/\hbar c) = (n/2)$ (Dirac, 1931). The latter point and more generally, the problem of the "monopole harmonics" will be reexamined and largely simplified on the basis of group theory considerations and of the analogy between a magnetic monopole and a symmetric top.

An important property of the equation suggested in this paper is that its $e^{i\gamma_5\theta}$ gauge invariance implies in a simple way the *symmetry laws* of the "free magnetism" deduced almost one century ago by Pierre Curie from the general laws of electricity and magnetism.

Just as in the neutrino theory and owing to the properties of our electromagnetic interaction, the equation splits into two equations describing a monopole-antimonopole pair. But the two particles differ from each other by their *opposite helicities* only, just as a neutrino-antineutrino pair does, whereas their charge g is the same: particles with an opposite g are not charge conjugated to the former and could be neither annihilated nor created by pairs with them. This property avoids the strong vacuum polarization which could be expected from the masslessness of these monopoles and will suggest the possible existence of an aether made of pairs of monopoles with opposite helicities, which should remain "invisible" to our ordinary means of observation.

The last part of the paper is devoted to a nonlinear generalization of the theory based on the same gauge invariance, which allows the introduction of a nonlinear term of mass. A parallel is made with previous works in which analogous nonlinear terms were found on the basis of a quite different hypothesis in the framework of general relativity. This leads us to the assertion that a magnetic monopole in an electromagnetic field "sees" a twisted space, the connection coefficients of which are to be identified with the pseudo-potential of the field.

2. THE TWO GAUGE INVARIANCES OF THE DIRAC EQUATION AND THE CORRESPONDING ELECTROMAGNETIC COUPLINGS

Let us first consider the equation of a free Dirac particle:

$$\gamma_\mu \partial_\mu \psi + \frac{m_0 c}{\hbar} \psi = 0 \tag{1}$$

in the relativistic coordinates $x_\mu = \{x_k, ict\}$, where the γ-matrices are (in terms of Pauli matrices s_k):

$$\gamma_k = i\begin{pmatrix} 0 & s_k \\ -s_k & 0 \end{pmatrix}, \qquad k = 1, 2, 3, \qquad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2}$$

We will prove that the equation (1) admits two and only two gauge transformations of the form:

$$\psi \to e^{i\Gamma\theta}\psi \tag{3}$$

where Γ is a constant Hermitian matrix and θ a constant parameter.

Introducing (3) in (1) we get:

$$(\gamma_\mu e^{i\Gamma\theta} \gamma_\mu)\gamma_\mu \partial_\mu \psi + \frac{m_0 c}{h} e^{i\Gamma\theta}\psi = 0 \tag{4}$$

But Γ may be written as:

$$\Gamma = \sum_{N=1}^{16} x_N \Gamma_N \tag{5}$$

with $\Gamma_N = \{I, \gamma_\mu, \gamma_{[\mu}\gamma_{\nu]}, \gamma_{[\lambda}\gamma_\mu\gamma_{\nu]}, \gamma_5\}$. Then, from the commutation rules of γ_μ, we have for any matrix Γ_N (Pauli, 1936):

$$\gamma_\mu \Gamma_N \gamma_\mu = \pm\Gamma_N \qquad (\mu = 1, 2, 3, 4;\ N = 1, 2, \ldots, 16) \tag{6}$$

where the $(\pm)$ sign varies according to μ and N.

Hence, for any Γ, we get:

$$\gamma_\mu e^{i\Gamma\theta} \gamma_\mu = \exp\left(i\theta \sum_{N=1}^{16} x_N \gamma_\mu \Gamma_\mu \gamma_\mu\right) = \exp\left(i\theta \sum_{N=1}^{16} (\pm) x_N \Gamma_N\right) \tag{7}$$

In order that (1) may be invariant under the transformation (3), a necessary condition is that $\gamma_\mu e^{i\Gamma\theta}\gamma_\mu$ does not depend on μ. But, among the sixteen matrices Γ_N there are only two that either commute with all the γ_μ (namely $\Gamma_N = I$), or anticommute (namely $\Gamma_N = \gamma_5$). The only possible form of Γ is thus:

$$\Gamma = x_1 I + x_2\gamma_5 \tag{8}$$

Actually, there are only two distinct cases (apart from a constant factor which will be absorbed in θ): (1) $\Gamma = I$, which corresponds to the classical phase invariance:

$$\psi \to e^{i\theta}\psi \tag{9}$$

(2) $\Gamma = \gamma_5$, which yields the *chiral gauge*:

$$\psi \to e^{i\gamma_5\theta}\psi \tag{10}$$

but while (9) is valid for any value of m_0, on the contrary (10) is valid only for $m_0 = 0$. The general case (8) has no special interest because it corresponds to the product of (9) and (10).

Let us now recall, for the sake of completeness, that (9) defines a minimal electromagnetic coupling by the covariant derivative:

$$\nabla_\mu = \partial_\mu - i\frac{e}{\hbar c}A_\mu \tag{11}$$

with the local gauge transformation:

$$\psi \to \exp\left(i\frac{e}{\hbar c}\phi\right)\psi, \qquad A_\mu \to A_\mu + \partial_\mu\phi \tag{12}$$

where the gauge field A_μ is a *polar* vector (the Lorentz potential), ϕ a *scalar* function and e an *electric* charge.

We shall follow a parallel procedure with the chiral gauge (10), introducing the covariant derivative:

$$\nabla_\mu = \partial_\mu - \frac{g}{\hbar c}\gamma_5 B_\mu \tag{13}$$

with the new local gauge transformation:

$$\psi \to \exp\left(i\frac{g}{\hbar c}\gamma_5\phi\right)\psi, \qquad B_\mu \to B_\mu + i\,\partial_\mu\phi \tag{14}$$

Because γ_5 is a *pseudo-scalar* operator, the gauge field B_μ will be now an *axial* vector, i.e., the dual of an antisymmetric tensor of rank three:

$$B_\mu = \overline{C_{\nu\rho\sigma}} = \frac{i}{3!}\varepsilon_{\mu\nu\rho\sigma}C^{\nu\rho\sigma} \tag{15}$$

in the same way, ϕ will now be a *pseudo*-scalar function, i.e, the unique component of a rank four antisymmetric tensor:

$$\phi = T_{1234} \tag{16}$$

(for these reasons, there is no factor i before B_μ in (13) whereas there is one before $\partial_\mu \phi$ in (14)).

In the following sections physical arguments will be given for the interpretation of B_μ as an electromagnetic *pseudo-potential* in the sense given by Cabibbo and Ferrari (Cabibbo and Ferrari, 1962) in the monopole theory. Therefore, g will be shown to be a *magnetic charge*. Let us recall that, in terms of B_μ, the electromagnetic field is given by:

$$F_{\mu\nu} = \overline{\partial_\mu B_\nu - \partial_\nu B_\mu} = i\varepsilon_{\mu\nu\rho\sigma}\, \partial^\rho B^\sigma = \partial^\rho C_{\mu\nu\rho} \tag{17}$$

and that Maxwell's equations in the presence of magnetic charges and in the absence of electric charges are:

$$\partial^\nu F_{\mu\nu} = 0, \qquad \partial^\nu \overline{F_{\mu\nu}} = \frac{4\pi}{c} K_\mu \tag{18}$$

K_μ is the magnetic current: it is an *axial* vector.

3. THE LINEAR EQUATION OF A MASSLESS SPIN $\frac{1}{2}$ MONOPOLE (SEE LOCHAK, 1983):

Our wave equation will be:

$$\gamma_\mu \left(\partial_\mu - \frac{g}{\hbar c} \gamma_5 B_\mu \right) \psi = 0 \tag{19}$$

Its interpretation as the equation of a monopole will be principally justified below by its application to the special case of a central electric field, but we have previously examined its symmetry properties.

(1) First of all, (19) is invariant under the gauge transformation (14). This entails the conservation of the *axial current*:

$$\partial_\mu K_\mu = 0, \qquad K_\mu = ig\bar{\psi}\gamma_\mu\gamma_5\psi \tag{20}$$

K_μ will play the role of a *magnetic current.* Its variance is in accordance with the Curie laws for the monopole symmetries (Curie, 1894) and moreover, this expression of K_μ was already suggested by Salam (Salam, 1966). The fact that this choice is imposed by the equation (19) is satisfactory.

Nevertheless, an intriguing difficulty arises about K_μ: it is found to be a *space-like* vctor. This property follows from the Darwin–de Broglie relations (Takabayasi, 1957):

$$-J_\mu J_\mu = \Sigma_\mu \Sigma_\mu = \Omega_1^2 + \Omega_2^2, \qquad J_\mu \Sigma_\mu = 0 \tag{21}$$

where J_μ, Σ_μ, Ω_1, Ω_2 are respectively the polar and axial vectors, the invariant and the pseudo-invariant defined by the Dirac spinor:

$$J_\mu = i\bar{\psi}\gamma_\mu\psi, \qquad \Sigma_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi, \qquad \Omega_1 = \bar{\psi}\psi, \quad \Omega_2 = -i\bar{\psi}\gamma_5\psi \tag{22}$$

We see from (21) that J_μ is a time-like vector, as is expected for a current density, but Σ_μ (and thus K_μ) is space-like, which seems unacceptable. But this objection will be removed in the next section.

It must be noted that the axial variance of the magnetic current K_μ is due to the fact that the magnetic charge is actually represented in (14) and (19) by a q-number, a pseudo-scalar charge operator:

$$G = g\gamma_5 \tag{23}$$

The pseudo-scalar property of a magnetic charge is well-known (see Curie, 1894), but it is very different to endow with such a symmetry the physical constant g itself, or to find it as a property of a quantum operator. The advantages of our representation will be shown below.

(2) Taking into account the axial variance of B_μ, one can verify that the equation (19) is P, T, C invariant, i.e., invariant under the three transformations:

$$P\colon\ x_k \to -x_k, \qquad x_4 \to x_4, \qquad B_k \to B_k, \qquad B_4 \to -B_4, \qquad \psi \to \gamma_4\psi \tag{24}$$

$$T\colon\ x_k \to x_k, \qquad x_4 \to -x_4, \qquad B_k \to -B_k, \qquad B_4 \to B_4, \qquad \psi \to \gamma_1\gamma_2\gamma_3\psi \tag{25}$$

$$C\colon\ g \to g, \qquad \psi \to \gamma_2\psi^* = \gamma_2\gamma_4\tilde{\bar{\psi}} \tag{26}$$

Note that the transformation (25) is the so-called "strong" time reversal. But the most important feature appears in the formula (26): the *charge conjugation does not change the sign of the magnetic charge* g. This means that a negative energy state is canonically transformed into a positive energy state of the *same particle* and not in a state of a particle with an opposite charge as it occurs for electrically charged particles. If we change g into $-g$ in the equation (19), we obtain a new particle, which is not conjugated to the former. Consequently, we cannot create or annihilate pairs of monopoles with opposite charges and we do not have to expect any magnetic vacuum polarization. This point is very important because if a vacuum polarization were possible with massless monopoles, we should have good reasons to fear it would be infinite or would at least give rise to enormous (and still never observed) physical effects.

However, we have now to answer the question: In what sense exactly can we speak of "charge conjugated" monopoles? The answer will be given by the representation of the equation (19) in which the operator G is diagonal.

4. THE TWO-COMPONENT THEORY AND THE ROLE OF ISOTROPIC CHIRAL CURRENTS

The operator G is diagonalized by the operator:

$$U = U^{-1} = \frac{1}{\sqrt{2}}(\gamma_4 + \gamma_5) \tag{27}$$

because

$$U\gamma_5 U^{-1} = \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{28}$$

The transformation $\psi \to U\psi$ defines the classical spinor representation of van der Waerden and we shall write:

$$U\psi = \frac{1}{\sqrt{2}}(\gamma_4 + \gamma_5)\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{29}$$

where ξ and η are *2-component* spinors which obey the eigenvalue equations:

$$UGU^{-1}\begin{pmatrix} \xi \\ 0 \end{pmatrix} = g\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \qquad UGU^{-1}\begin{pmatrix} 0 \\ \eta \end{pmatrix} = -g\begin{pmatrix} 0 \\ \eta \end{pmatrix} \tag{30}$$

Now, introducing (29) in the equation (19), taking (2) into account and dropping the relativistic notations, our equation splits into the following system (Lochak, 1983):

$$\begin{aligned} \left(\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}\cdot\nabla - i\frac{g}{\hbar c}(W + \mathbf{s}\cdot\mathbf{B})\right)\xi = 0 \\ \left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}\cdot\nabla + i\frac{g}{\hbar c}(W - \mathbf{s}\cdot\mathbf{B})\right)\eta = 0 \end{aligned} \tag{31}$$

where $\mathbf{s}$ represents the Pauli matrices and

$$iB_\mu = \{\mathbf{B}, iW\} \tag{32}$$

Note that because B_μ is axial, its space components are imaginary and B_4 is real: the converse would be true for a polar vector.

In absence of external potential B_μ, the equations (31) reduce to the 2-component neutrino equations and we see that even in the presence of a potential B_μ, equation (19) describes in fact a *couple of monopoles* represented by (31): (a) A left monopole ξ with helicity $-\frac{1}{2}$ corresponding to the eigenvalue $+g$ of the charge operator G; (b) A right monopole η with helicity $+\frac{1}{2}$ corresponding to the eigenvalue $-g$ of G.

The two equations (31)—and thus the two particles—exchange under the transformations P, T, C:

$$P:\ t\to t,\qquad \mathbf{x}\to-\mathbf{x},\qquad W\to -W,\qquad \mathbf{B}\to\mathbf{B},\qquad \xi\leftrightarrow\eta \tag{33}$$

$$T:\ t\to -t,\qquad \mathbf{x}\to\mathbf{x},\qquad W\to W,\qquad \mathbf{B}\to-\mathbf{B},\qquad \xi\leftrightarrow\eta \tag{34}$$

$$C:\ g\to g,\qquad -is_2\xi^*\to\eta,\ is_2\eta^*\to\xi \tag{35}$$

In these transformations, we must remember that W is a pseudo-scalar and $\mathbf{B}$ a pseudo-vector. It follows from these transformations, that each equation (31) is PT, CP, and CT invariant.

The transformation (35) may be deduced from (26) and (29). It shows that the two monopoles ξ and η are charge conjugated, but although they correspond respectively to the eigenvalues $+g$ and $-g$ of the charge operator G, we see again that the *sign of the constant g does not change* in the C transformation. In fact the essential difference between ξ and η is the *sign of the helicity*, just like between the neutrino and the anti-neutrino and in accordance with the symmetry laws of Curie.

If we change the sign of g in the system (31), we obtain two different equations which are irreducible to the previous system by any unitary transformation:

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{s}\cdot\nabla+i\frac{g}{\hbar c}(W+\mathbf{s}\cdot\mathbf{B})\right)\xi'=0\\ \left(\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{s}\cdot\nabla-i\frac{g}{\hbar c}(W-\mathbf{s}\cdot\mathbf{B})\right)\eta'=0\end{aligned} \tag{31a}$$

The systems (31) and (31a) describe *two* different pairs of monopoles. The two particles of each pair differ by their helicities and the two pairs by the sign of the constant g.

The equations (31) are invariant under the gauge transformation:

$$\xi\to e^{i(g/\hbar c)\phi}\xi,\qquad \eta\to e^{-i(g/\hbar c)\phi}\eta,\qquad W\to W+\frac{1}{c}\frac{\partial\phi}{\partial t},\qquad \mathbf{B}\to\mathbf{B}-\nabla\phi \tag{36}$$

Note that the chiral gauge corresponds here to phase transformations with opposite signs for ξ and η, in accordance with the opposite helicities of ξ and η and with the fact that ϕ is a pseudo-scalar.

In virtue of the gauge law (36) the system (31)—just as (31a)—entails the conservation of two chiral currents:

$$\frac{\partial(\xi^+\xi)}{\partial t}-c\nabla(\xi^+\mathbf{s}\xi)=0,\qquad \frac{\partial(\eta^+\eta)}{\partial t}+c\nabla(\eta^+\mathbf{s}\eta)=0 \tag{37}$$

Defining the two currents as:

$$X_\mu = \{\xi^+\xi, -\xi^+\mathbf{s}\xi\}, \qquad Y_\mu = \{\eta^+\eta, \eta^+\mathbf{s}\eta\} \tag{38}$$

and making use of the transformation (29), we find the following decomposition of the polar and axial vectors (22):

$$J_\mu = X_\mu + Y_\mu, \qquad \Sigma_\mu = X_\mu - Y_\mu \tag{39}$$

This means, in particular, that our *magnetic current* $K_\mu = g\Sigma_\mu$, given by (20), is equal to the *difference* of the two magnetic currents gX_μ and gY_μ respectively associated to the left and right monopoles. Such a difference does not have to be of a definite type and the fact that K_μ is space-like is not shocking any more: this is the answer to the question raised in Section 3. Moreover, one can easily prove from (38) that the currents X_μ and Y_μ are *isotropic*:

$$X_\mu X_\mu = 0, \qquad Y_\mu Y_\mu = 0 \tag{40}$$

which is in accordance with the fact that our monopoles are massless.

Finally, we can verify on (37) and (38) that the parity transformation involves an exchange between X_μ and Y_μ:

$$\mathbf{x} \to -\mathbf{x} \Rightarrow X_\mu \leftrightarrow Y_\mu \tag{41}$$

which justifies their denomination as *chiral* currents.

5. THE MONOPOLE IN A CENTRAL ELECTRIC FIELD. THE PROBLEM OF "MONOPOLE HARMONICS"

In order to solve the central field problem, we first have to find the expressions of W and $\mathbf{B}$ for a central electric field. From (17) and (32) we have:

$$\operatorname{rot} \mathbf{B} = \mathbf{E} = e\frac{\mathbf{r}}{r^3}, \qquad W = 0 \tag{42}$$

As we have said in Section 1, we will choose neither the representation of Wu and Yang in terms of a connection on a fiber bundle, nor exactly the Dirac solution:

$$B'_x = \frac{e}{r}\frac{-y}{r+z}, \qquad B'_y = \frac{e}{r}\frac{x}{r+z}, \qquad B'_z = 0, \qquad (r = \sqrt{x^2+y^2+z^2}) \tag{43}$$

We choose instead:

$$B_x = \frac{e}{r}\frac{yz}{x^2+y^2}, \qquad B_y = \frac{e}{r}\frac{-xz}{x^2+y^2}, \qquad B_z = 0 \tag{44}$$

which differs from the former by a simple choice of gauge:

$$\mathbf{B}-\mathbf{B}'=\nabla \text{ Arctg} \frac{y}{x} \tag{45}$$

But the true difference is that (44) is an *axial vector*, as $\mathbf{B}$ must be, while the solution (43) has *no definite parity*. Some simplifications will occur from this choice in the computations.

In polar coordinates ($x=r\sin\theta\cos\Phi$, $y=r\sin\theta\sin\Phi$, $z=\cos\theta$), we have:

$$B_x=-\frac{e\sin\Phi}{r\ \text{tg}\,\theta},\qquad B_y=-\frac{e\cos\Phi}{r\ \text{tg}\,\theta},\qquad B_z=0 \tag{46}$$

Introducing (44) in the 2-component equations (31), one can verify that the latter admit respectively the following left and right Poincaré integrals (Poincaré, 1896) of motion, corresponding to the monopole and to the antimonopole:

$$\mathbf{J}_\xi=\hbar[\mathbf{r}X(-i\nabla+D\mathscr{B})+D\hat{r}+\tfrac{1}{2}\mathbf{s}] \tag{47}$$

$$\mathbf{J}_\eta=\hbar[\mathbf{r}X(-i\nabla-D\mathscr{B})-D\hat{r}+\tfrac{1}{2}\mathbf{s}] \tag{48}$$

with the notations:

$$D=\frac{eg}{\hbar c},\qquad \mathbf{B}=e\mathscr{B},\ \hat{r}=\frac{\mathbf{r}}{r} \tag{49}$$

We know that the *Dirac number* D will play a fundamental role. $\mathbf{J}_\xi$ and $\mathbf{J}_\eta$ differ only by the sign of D: we can thus restrict our study to the first equation (31) and consider only $\mathbf{J}_\xi$, dropping the subscript ξ. Let us recall that $\mathbf{J}$ is a total angular momentum: $\hbar\mathbf{r}\times(-i\nabla+D\mathscr{B})$ is the orbital part, $hD\hat{r}$ corresponds to the external field (Thomson, 1904; Goldhaber, 1965), $\hbar/2\,\mathbf{s}$ is the spin. The components of $\mathbf{J}$ obey the relations:

$$[J_2,J_3]=i\hbar J_1,\qquad [J_3,J_1]=i\hbar J_2,\qquad [J_1,J_2]=i\hbar J_3 \tag{50}$$

The angular part of the stationary solutions of the first equation (31) (they will be the "monopole harmonics") are the eigenstates of J^2 and we can choose those which are also eigenstates of J_3. We will give for this problem a procedure which considerably simplifies the calculations.

Let us first write $\mathbf{J}$ in the form:

$$\mathbf{J}=\hbar(\mathbf{\Lambda}+\tfrac{1}{2}\mathbf{s}),\qquad \mathbf{\Lambda}=\mathbf{r}\times(-i\nabla+D\mathscr{B})+D\hat{r} \tag{51}$$

The components of $\mathbf{\Lambda}$ satisfy the relations:

$$[\Lambda_2,\Lambda_3]=i\Lambda_1,\qquad [\Lambda_3,\Lambda_1]=i\Lambda_2,\qquad [\Lambda_1,\Lambda_2]=i\Lambda_3 \tag{52}$$

and they have the following expressions in terms of polar angles:

$$\Lambda^{+} = \Lambda_1 + i\Lambda_2 = e^{i\Phi}\left(i \cot\theta \frac{\partial}{\partial\Phi} + \frac{\partial}{\partial\theta} + \frac{D}{\sin\theta}\right)$$

$$\Lambda^{-} = \Lambda_1 - i\Lambda_2 = e^{-i\Phi}\left(i \cot\theta \frac{\partial}{\partial\Phi} - \frac{\partial}{\partial\theta} + \frac{D}{\sin\theta}\right) \tag{53}$$

$$\Lambda_3 = -i\frac{\partial}{\partial\Phi}$$

Notice that, owing to the choice of the solutions (44) for **B** we have *no additional term in* Λ_3, as occurs with the Dirac solution (43) (Wu and Yang, 1975, 1976). To solve our problem we must find the eigenstates $Z(\theta, \Phi)$ of $\mathbf{\Lambda}^2$ and Λ_3:

$$\mathbf{\Lambda}^2 Z = j(j+1)Z, \qquad \Lambda_3 Z = mZ \tag{54}$$

with

$$j = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots, \qquad m = -j, -j+1, \ldots, j-1, j \tag{55}$$

These eigenvalues are known a priori as a consequence of the commutation relations (52). But instead of computing $Z(\theta, \Phi)$, we shall introduce a third angle χ (the meaning of which will soon appear) and we shall consider the product:

$$\mathcal{D}(\theta, \Phi, \chi) = e^{iD\chi} Z(\theta, \Phi) \tag{56}$$

The functions $\mathcal{D}(\theta, \Phi, \chi)$ will be eigenstates of new operators $\mathbf{R}^2$ and R_3 with the same eigenvalues as Z:

$$\mathbf{R}^2 \mathcal{D} = j(j+1)\mathcal{D}, \qquad R_3\mathcal{D} = m\mathcal{D} \tag{57}$$

The new set of operators R_1, R_2, R_3 is immediately derived from (53)

$$R^{+} = R_1 + iR_2 = e^{i\Phi}\left(i \cot\theta \frac{\partial}{\partial\Phi} + \frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\chi}\right)$$

$$R^{-} = R_1 = iR_2 = e^{-i\Phi}\left(i \cot\theta \frac{\partial}{\partial\Phi} - \frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\chi}\right) \tag{58}$$

$$R_3 = -i\frac{\partial}{\partial\Phi}$$

But these operators are well-known: they are the *infinitesimal operators of the rotation group* in terms of the Euler angles: θ is the angle of *nutation*, Φ is the *precession*, and the angle χ we have just introduced is the *proper rotation* angle. The operators R_k are expressed in the *fixed referential*: the more common expressions in the *moving referential* will be denoted Q_k and we have (Lochak, 1959; Gelfand, 1956, 1963):

$$Q_k(\theta, \Phi, \chi) = R_k(\theta, \pi - \chi, \pi - \Phi) \tag{59}$$

Is it in any way surprising to come across these operators R_k (or Q_k)? Certainly not, because the dynamical system constituted by a monopole in a central electric field has the *spherical symmetry* and a general theorem (Gelfand, 1956, 1963) claims that when such a system is described by a linear differential equation (as it is in our case) a general continuous solution may be expanded in a series of *generalized spherical functions*, i.e., the matrix elements of the linear representations of the rotation group. They are precisely the solutions of the problem (57).

This is why the orbital angular momentum Λ given by (53) could not be anything else but the infinitesimal rotation operator, just like for a symmetric *top* (Lochak, 1959): the only relation between the motion of a top around a fixed point and the motion of a monopole around a fixed electric charge (or conversely the motion of an electric charge around a fixed massive monopole) is the *invariance under the rotation group.* This explains why the angular functions in both cases are identical. The fact itself has been known for a long time (Tamm, 1931; Fierz, 1944), but this simple explanation does not seem to have been given before.

Under the only condition of *continuity* on the rotation group, the solution of the eigenstates problem (57) is:

$$\mathscr{D}_j^{m',m}(\theta, \Phi, \chi) = e^{i(m\Phi + m'\chi)} d_j^{m',m}(\theta) \tag{60}$$

$$d_j^{m',m}(\theta) = N(1-u)^{-(m-m')/2}(1+u)^{-(m+m')/2}\left(\frac{d}{du}\right)^{j-m}[(1-u)^{j-m'}(1+u)^{j+m'}] \tag{61}$$

$$u = \cos\theta, \qquad N = \frac{(-1)^{j-m} i^{m-m'}}{2^j}\left(\frac{(j+m)!}{(j-m)!(j-m')!(j+m')!}\right)^{1/2} \tag{62}$$

$$j = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots, \text{etc.} \ldots, \qquad m, m' = -j, -j+1, \ldots, j-1, j \tag{63}$$

These formulae are given in all the textbooks on group theory. The constant N in $\mathscr{D}_j^{m'm}$ is so defined that the rows (and the columns) of the *unitary* $(2j+1)$-matrix of the representation $\mathscr{D}_j$ are normed to unity. Hence, if we go back to the *normalized angular quantum states* $Z(\theta, \Phi)$ (56), we find the "monopole harmonics":

$$Z_j^{m',m}(\theta, \Phi) = \sqrt{2j+1}\,\mathscr{D}_j^{m'm}(\theta, \Phi, 0) \tag{64}$$

The angle χ of proper rotation disappears from the eigenstate of the monopole because the latter is *punctual* in this theory: the existence of a projection $\hbar m' \neq 0$ of the orbital angular momentum on the *symmetry axis* of the system is only due to the chiral properties of the magnetic charge and to the angular momentum thus induced on the electromagnetic field, but there is no spatial extension of the monopole, contrary to the case of a symmetric top.

However, a crucial point is, precisely, the expression $\hbar m'$ of this projection, because we get from (56) and (60):

$$D = m' \tag{65}$$

Then, *the Dirac number is the proper rotation quantum number.* Its values are given by (63) and, from (49), we get the famous Dirac law (Dirac, 1931):

$$\frac{eg}{\hbar c} = \frac{n}{2} \,(n\text{: integer}) \tag{66}$$

It must be emphasized that this relation is deduced from the sole hypothesis of the *continuity* of the matrix elements $\mathcal{D}_j^{m',m}$ (and therefore of the eigenstates $Z_j^{m',m}$) on the rotation group: no other condition is necessary to solve the problem (57) (Lochak, 1959, 1984).

It is perhaps not useless to underline that the functions (60) and (61) are really continuous for *all* the values of θ, m, and m' and that, despite their appearance, they do not have any "mark" of the discontinuous string of the pseudo potential **B** (46). This continuity may be proved directly by a property of the Jacobi polynomials, but actually it lies in the very definition of the "generalized spherical functions" $\mathcal{D}_j^{m',m}$.

Finally, by the Clebsch–Gordan procedure, we find the harmonics with spin:

$$\begin{aligned} \Omega_j^{m',m_{(+)}} &= \begin{pmatrix} \left(\dfrac{j+m}{2j+1}\right)^{1/2} & Z_j^{m',m-1} \\ \left(\dfrac{j-m+1}{2j+1}\right)^{1/2} & Z_j^{m',m} \end{pmatrix}, \\ \Omega_j^{m',m_{(-)}} &= \begin{pmatrix} \left(\dfrac{j-m+1}{2j+1}\right)^{1/2} & Z_j^{m',m-1} \\ -\left(\dfrac{j+m}{2j+1}\right)^{1/2} & Z_j^{m',m} \end{pmatrix} \end{aligned} \tag{67}$$

further denoted Ω_j^+ and Ω_j^- by abbreviation and which correspond respectively to the eigenvalues $k=j+\frac{1}{2}$ and $k=j-\frac{1}{2}$ of $\mathbf{J}^2$. With $k=j-\frac{1}{2}$, we have for instance:

$$\mathbf{J}^2\Omega_{j-1}^+ = \hbar^2 k(k+1)\Omega_{j-1}^+, \quad \mathbf{J}^2\Omega_j^- = \hbar^2 k(k+1)\Omega_j^- \tag{68}$$

In order to achieve the integration it will be useful to make the operator $\mathbf{s}\cdot\hat{r}$ act on the spinors $\Omega_j^\pm$. The following formulae are deduced from the recurrence relations between the generalized spherical functions (Gelfand, 1963):

$$\begin{aligned} \mathbf{s}\cdot\hat{r}\Omega_{j-1}^+ &= \cos\theta'\Omega_{j-1}^+ + \sin\theta'\Omega_j^- \\ \mathbf{s}\cdot\mathbf{r}\Omega_j &= \sin\theta'\Omega_{j-1}^+ - \cos\theta'\Omega_j^- \end{aligned} \tag{69}$$

$$\cos\theta' = \frac{m'}{j} = \frac{D}{j} \tag{70}$$

The angle θ' has a double physical meaning at the geometrical optics approximation:

(1) $\hbar j$ is the total orbital momentum (without spin) and $\hbar D$ the angular momentum of the external field. Therefore, θ' is the angle between these to momenta: it is thus the *vertex half-angle of the Poincaré cone* (Poincaré, 1896; Goldhaber, 1965), the geodesics of which are the trajectories of a monopole in an electric central field (or of an electric charge around a fixed monopole).

(2) But on the other hand, $\hbar m'$ is the projection of the total orbital momentum on the *symmetry axis* of the system: hence, θ' may also be defined as the *vertex half-angle of the cone enveloped by the precession of the symmetry axis around the total angular momentum.* This new definition of the Poincaré cone exactly coincides with the definition of the *Poinsot cone* of a symmetric top. The identity between these two cones (of Poincaré and Poinsot) is another consequence of the identity of both problems (the monopole and the top) with regard to the invariance under the rotation group. Note that the same angle θ' appears in the geometrical optics approximation of a quantum top (Lochak, 1959).

6. THE RADIAL FUNCTIONS OF THE MONOPOLE

Introducing (49) and (61) in the ξ-equation (31), we find:

$$\frac{i}{c}\frac{\partial\xi}{\partial t} = \mathbf{s}\cdot(i\nabla - m'\mathcal{B})\xi \tag{71}$$

where $W=0$ and $\mathcal{B} = 1/e\,\mathbf{B}$ is given by (44) or (46). A solution ξ with a given total angular momentum $k = j - \frac{1}{2}$ will take the form:

$$\xi = e^{-i\omega t}(F_{j-1}^+(r)\Omega_{j-1}^+ + F_j^-(r)\Omega_j^-) \tag{72}$$

where $\Omega^{\pm}$ are the angular spinors (67) and $F^{\pm}(r)$ are radial functions we now have to compute. To this end, we shall apply a classical integration method of the hydrogen atom in Dirac's theory. Introducing (72) in (71) and multiplying by $\mathbf{s}\cdot\hat{r}$ we get:

$$\frac{\omega}{c}\mathbf{s}\cdot\hat{r}(F_{j-1}^{+}\Omega_{j-1}^{+}+F_j^{-}\Omega_j^{-})=\mathbf{s}\cdot\hat{r}\mathbf{s}\cdot(i\nabla-m'\mathscr{B})(F_{j-1}^{+}\Omega_{j-1}^{+}+F_j^{-}\Omega_j^{-}) \tag{73}$$

Using (44), (51) and the classical algebraic relation:

$$\mathbf{s}\cdot\mathbf{A}_1\mathbf{s}\cdot\mathbf{A}_2=\mathbf{A}_1\cdot\mathbf{A}_2+i(\mathbf{A}_1\times\mathbf{A}_2)\cdot\mathbf{s} \tag{74}$$

the equation (73) takes the form:

$$\frac{dF_{j-1}^{+}}{dr}\Omega_{j-1}^{+}+\frac{dF_j^{-}}{dr}\Omega_j^{-}=\frac{1}{r}\mathbf{\Lambda}\cdot\mathbf{s}(F_{j-1}^{+}\Omega_{j-1}^{+}+F_j^{-}\Omega_j^{-})$$
$$-\left(\frac{m'}{r}+i\frac{\omega}{c}\right)\mathbf{s}\cdot\hat{r}(F_{j-1}^{+}\Omega_{j-1}^{+}+F_j^{-}\Omega_j^{-}) \tag{75}$$

We know that $\Omega^{\pm}$ are eigenvectors of $\mathbf{J}^2$ and $\mathbf{\Lambda}^2$ with eigenvalues $h^2k(k+1)$ and $h^2j(j+1)$; we have then (for $k=j-\frac{1}{2}$):

$$\mathbf{\Lambda}\cdot\mathbf{s}\Omega_{j-1}^{+}=(j-1)\Omega_{j-1}^{+},\qquad \mathbf{\Lambda}\cdot\mathbf{s}\Omega_j^{-}=-(j+1)\Omega_j^{-} \tag{76}$$

It is easy to eliminate $\Omega^{\pm}$ from (75), multiplying on the left successively by Ω_{j-1}^{+} and by Ω_j^{-} and integrating on the angles. Making use of (69), we find:

$$\left(\frac{d}{dr}+\frac{1}{r}-\frac{j}{r}s_3+\left(\frac{m'}{r}+i\frac{\omega}{c}\right)e^{-is_2(\theta'/2)}s_3\,e^{is_2(\theta'/2)}\right)F=0 \tag{77}$$

$$F(r)=\begin{pmatrix}F_{j-1}^{+} & (r)\\ F_j^{-} & (r)\end{pmatrix} \tag{78}$$

Introducing new functions $G_{j-1}^{+}(r)$ and $G_j^{-}(r)$ such that:

$$F=\frac{1}{r}\exp\left[is_2\left(\frac{\pi}{4}-\frac{\theta'}{2}\right)\right]G,\qquad G=\begin{pmatrix}G_{j-1}^{+}\\ G_j^{-}\end{pmatrix} \tag{79}$$

the equation (77) becomes:

$$\left(\frac{d}{dr}-\frac{l}{r}s_3+i\frac{\omega}{c}s_1\right)G=0 \tag{80}$$

where $l\geq 0$ has the value:

$$l=j\sin\theta'=(j^2-m'^2)^{1/2} \tag{81}$$

l is the projection of the total orbital angular momentum (monopole + field) on the symmetry plane orthogonal to the axis of the Poincaré cone.

Differentiating (80), we get two Bessel equations (Ince, 1956) for the components of G:

$$\frac{d^2G_{j-1}^+}{dr}+\left(\frac{\omega^2}{c^2}-\frac{l(l-1)}{r^2}\right)G_{j-1}^+=0, \qquad \frac{d^2G_j^-}{dr}+\left(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2}\right)G_j^-=0 \tag{82}$$

Taking into account (80) and the recurrence formula:

$$zJ'_\lambda(z)+\lambda J_\lambda(z)=zJ_{\lambda-1}(z) \tag{83}$$

we find the solution:

$$G=\left(r\frac{\omega}{c}\right)^{1/2}\begin{pmatrix} iJ_{l-1/2}\left(\frac{\omega}{2}r\right) \\ J_{l+1/2}\left(\frac{\omega}{2}r\right)\end{pmatrix} \tag{84}$$

Inserting this result in (79) and then in (72), we obtain the ξ functions. We would get, by a similar procedure, the spinor η.

In conclusion of the last two sections, we can say that: (a) Our equations (31) give the correct expressions (47) and (48) for the angular momentum of a monopole in a coulombian field. (b) The Dirac relation (66) is deduced in a simple way from general conditions of rotation invariance and continuity of the wave functions. (c) The geometrical analogy with the classical corresponding problem is made evident in the case of the Coulomb interaction. (d) The radial functions (84) are exactly the same as the corresponding ones for an electrically charged massless fermion in the field of an infinitely massive monopole: see for example formulae (23) in Kazama (Kazama, 1977) with $M=0$ and note that equation (22) of the same reference (for $M=0$) corresponds exactly to the above equation (80) (our l is denoted μ in Kazama (Kazama, 1977)).

We shall not discuss further the properties of these radial functions and, especially, we shall not enter into the subtleties of the lowest possible level of the angular momentum thoroughly examined in several papers (Kazama, 1977; Yamagishi, 1983).

We shall try to answer another question. Until now, we could ask: "Does the system (31) actually represent monopoles?": the results of the last two sections give a positive answer. But now, from the close analogy between our results and the classical ones, arises a new question: "What is new in this theory? What is the difference between this theory and the one which would be more directly obtained if we had tried to describe a spin $\frac{1}{2}$ monopole by simply introducing a pseudo-scalar constant of charge g in the Dirac equation?".

7. MUST THE MAGNETIC CHARGE BE A PSEUDO-SCALAR c-NUMBER g OR A PSEUDO-SCALAR q-NUMBER $g\gamma_5$ (WITH SCALAR g)?

In order to make the comparison easier, we shall consider the case of a massless monopole, but instead of the equation (19) we write:

$$\gamma_\mu \left(\partial_\mu - \frac{g}{\hbar c} B_\mu \right) \psi = 0 \tag{85}$$

We have introduced the same quasi-potential B_μ as in (19) (there is no factor i as there would be before A_μ in the usual Dirac equation, because B_μ is *axial*). The charge operator is no more $G = g\gamma_5$, like in (19), but:

$$G' = gI \tag{86}$$

so that (85) is no longer invariant under the chiral gauge transformation (14), but only under the ordinary phase translation:

$$\psi \to e^{i(g/\hbar c)\Phi\psi}, \; B_\mu \to B_\mu + i\partial_\mu \Phi \tag{87}$$

Hence, the associated conservative current is not the axial vector Σ_μ but the usual polar one J_μ and instead of (20) we have:

$$K'_\mu = gJ_\mu = ig\bar{\psi}\gamma_\mu\psi \tag{88}$$

Since J_μ is polar and B_μ axial, the equation (85) will be P and T invariant only if we admit that the *c-number g itself is a pseudo-scalar.*

To compare this new theory with our previous one, the best thing to do is to write the 2-component representation:

$$\begin{aligned} \left(\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s} \cdot \nabla - i\frac{g}{\hbar c}(W + \mathbf{s} \cdot \mathbf{B}) \right) \xi = 0 \\ \left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s} \cdot \nabla - i\frac{g}{\hbar c}(W - \mathbf{s} \cdot \mathbf{B}) \right) \eta = 0 \end{aligned} \tag{89}$$

The difference with (31) seems to be very small: only the *sign* before i in the second equation. But the consequences are very important: whereas the ξ and η equations exchange between themselves in (89)—just like in (31)—under the P and T transformations (33) and (34) (because g is supposed to be a pseudo-scalar), on the contrary the *charge conjugation* is no more (35) but:

$$C: g \to -g, \qquad -is_2\xi^* \to \eta, \qquad is_2\eta^* \to \xi \tag{90}$$

The monopole and the anti-monopole are not only left and right: they have moreover *opposite charges.* Monopoles with opposite charges can now be created or annihilated by pairs, involving (in the case of a zero, or weak mass) a strong vacuum polarization which has never been observed.

However, with the phase transformation (87) instead of the chiral gauge transformation (21), we are no more confined to the massless case, so that we could introduce in the equation (85) a mass term, that would give cross terms in (89), which could weaken the vacuum polarization, if the monopole is taken sufficiently heavy. But the principal objection against such a theory does not lie here: it lies in the fact that the particle described by (85) is not a true monopole, but rather an electrically charged particle which is as to say "disguised" into a monopole.

Actually we can introduce a dimensionless pseudo-scalar S, such that $S^2 = 1$ and define the following quantities:

$$e = Sg, \qquad A_\mu = iSB_\mu, \qquad J_\mu = SK'_\mu \tag{91}$$

Introducing (91) in (85), we cannot distinguish this equation from the equation of an electrically charged massless fermion. And, adding a term of mass, we find the usual Dirac equation! We are faced with a particular case of a classical objection against the monopole theory (Cabrera and Trower, 1983; Jackson, 1975; Harrison et al, 1963). It was argued that if we generalize Maxwell's equations by adding densities of magnetic charge and current:

$$\begin{aligned} &\operatorname{rot} \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 4\frac{\pi}{c}\mathbf{J} \qquad && -\operatorname{rot} \mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 4\pi \mathbf{K} \\ &\operatorname{div} \mathbf{E} = 4\pi\rho, && \operatorname{div} \mathbf{H} = 4\pi\mu \end{aligned} \tag{92}$$

the system thus obtained is invariant under the following transformation:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}'\cos\gamma + \mathbf{H}'\sin\gamma, \qquad & \mathbf{H} &= -\mathbf{E}'\sin\gamma + \mathbf{H}'\cos\gamma \\ \rho &= \rho'\cos\gamma + \mu'\sin\gamma, & \mu &= -\rho'\sin\gamma + \mu'\cos\gamma \\ \mathbf{J} &= \mathbf{J}'\cos\gamma + \mathbf{K}'\sin\gamma, & \mathbf{K} &= -\mathbf{J}'\sin\gamma + \mathbf{K}'\cos\gamma \end{aligned} \tag{93}$$

where γ is an arbitrary constant pseudo-scalar angle. Thus, we can (so goes the argument) eliminate the magnetic terms by a convenient choice of γ and go back to the usual Maxwell equations: consequently the problem of monopole would be a simple question of terminology, not a physical problem. Actually, we shall see that this reasoning is false in the case of our theory but it is true for the one which is examined in this section: the transformation (91) simply corresponds to introducing in (93) a value of γ such that

$$\sin\gamma = S, \qquad \cos\gamma = 0 \tag{94}$$

from which follows the transformation of a purely magnetic particle in a purely electric one. In this sense, we can assert that the equation (85) does not represent a "true" monopole but rather a "disguised" electric charge and reject, for this reason, the all too simple idea of representing a magnetic charge by a pseudo-scalar c-number.

Our theory, based on the charged operator $g\gamma_5$, i.e., on a q-number, is completely different, because there, it is impossible to reduce magnetism to electricity using a transformation like (91) or (93). To prove this assertion, it is enough to observe that current densities of electricity (J_μ) and magnetism (K_μ) *cannot be proportional*, for the elementary reason that J_μ is time-like and K_μ space-like. In another way, we could ask the question: is it possible to find a canonical transformation of (19), i.e., a unitary matrix U, such that J_μ would be transformed in Σ_μ? It would be so only if:

$$U\gamma_5 U^+ = qI, \qquad (U^+U = I) \tag{95}$$

but this is impossible because:

$$\mathrm{tr}\ \gamma_5 = 0 \tag{96}$$

(remember that (96) is an invariant property (Pauli, 1936) implied by the anticommutation of Dirac's matrices).

The electromagnetic coupling we have introduced in the equation (19) is thus irreducible to the coupling appearing in the usual Dirac equation of the electron and we may assert that it actually describes an "independent" particle, different from the electron.

The fact that there are two and only two electromagnetic gauges in the Dirac equation and that one of them corresponds to an electric charge and the other one to a magnetic monopole is striking. If we think of the heuristic power of the Dirac equation, it is difficult not to believe in the existence of this monopole.

The equation (19) is the only possible one in the framework of the Dirac formalism, but only in the linear case. We will now examine nonlinear possibilities, but we need previously some properties of the chiral gauge.

8. TRANSFORMATION OF THE DIRAC TENSORS UNDER THE CHIRAL GAUGE

For the sake of the simplification of some formulae, we shall write (10) in the form:

$$\psi' = e^{i\gamma_5(\theta/2)}\psi \tag{97}$$

Introducing this formula in the 16 expressions:

$$\Omega_1 = \bar{\psi}\psi, \qquad J_\mu = i\bar{\psi}\gamma_\mu\psi, \qquad \Sigma_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi, \qquad M_{\mu\nu} = \bar{\psi}\gamma_{[\mu}\gamma_{\nu]}\psi, \qquad \Omega_2 = -i\bar{\psi}\gamma_5\psi \tag{98}$$

and denoting with primes the same quantities transformed by (97), we find:

$$J'_\mu = J_\mu, \qquad \Sigma'_\mu = \Sigma_\mu \tag{99}$$

$$\begin{pmatrix} \Omega'_1 \\ \Omega'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \tag{100}$$

$$\begin{pmatrix} \bar{M}'_{\mu\nu} \\ M'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{M}_{\mu\nu} \\ M_{\mu\nu} \end{pmatrix} \tag{101}$$

$\bar{M}_{\mu\nu}$ denoting the dual tensor of $M_{\mu\nu}$. In fact, (101) is also a consequence of (100) and of the Yvon-Kofinck formula (Takabayasi, 1957; Yvon, 1940):

$$(\Omega_1^2 + \Omega_2^2) M_{\mu\nu} = \Omega_1 \overline{(J_\mu \Sigma_\nu - J_\nu \Sigma_\mu)} - \Omega_2 (J_\mu \Sigma_\nu - J_\nu \Sigma_\mu) \tag{102}$$

The most interesting formula is (100) which says that in the "chiral plane" the "vector" (Ω_1, Ω_2) undergoes a rotation of an angle θ.

It could seem, looking at the relations (99), (100), and (101) that we can define a relatively great number of quantities, invariant under the Lorentz and chiral transformations. Actually, taking into account the fact that the tensor quantities are connected between them through (21) and (102), only *one* independent invariant remains, the length of the vector (Ω_1, Ω_2) in the chiral plane:

$$\rho = (\Omega_1^2 + \Omega_2^2)^{1/2} \tag{103}$$

It is convenient to introduce a *pseudo-scalar* angle A in the chiral plane, such that:

$$\Omega_1 = \rho \cos A, \qquad \Omega_2 = \rho \sin A \tag{104}$$

Then, the transformations (97) and (100) both reduce to:

$$A' = A + \theta \tag{105}$$

It is worth mentioning that the angle A plays an important—but still mysterious—role in the Dirac equation (Takabayasi, 1957; Jakobi and Lochak, 1956a, 1956b; Lochak, 1957; Halbwacks, 1960). If a quantity is a chiral invariant, it does not depend on A. If it is, in addition, Lorentz invariant, it depends only on ρ.

Then, we shall consider the Lagrangian density:

$$L = \tfrac{1}{2} \bar{\psi} \gamma_\mu [\partial_\mu] \psi - \frac{g}{\hbar c} \bar{\psi} \gamma_\mu \gamma_5 B_\mu \psi + \frac{1}{4} \frac{\mathcal{M}(\rho^2) c}{\hbar} \tag{106}$$

where ρ is the quantity (103) and $\mathcal{M}(\rho^2)$ is an arbitrary real scalar function which has the dimension of a mass.

9. A NONLINEAR EQUATION FOR A SPIN $\frac{1}{2}$ MONOPOLE

From the Lagrangian (106), we derive the field equation (Lochak, 1983):

$$\left(\gamma_\mu\left(\partial_\mu-\frac{g}{\hbar c}\gamma_5 B_\mu\right)+\frac{1}{2}\frac{m(\rho^2)c}{\hbar}(\Omega_1-i\Omega_2\gamma_5)\right)\psi=0 \qquad (107)$$

where $m(\rho^2)$ is the derivative of the function $\mathcal{M}(\rho^2)$ introduced in the lagrangian.

From its very definition, the equation (107) is invariant under the chiral gauge transformation (14) and the magnetic current K_μ is still conserved as it was for the linear equation (19). Likewise, the equation is *P-invariant*: one can prove it easily with the transformation (24). But on the contrary, one can see, using (25) and (26), that the equation (107) is neither T nor C-invariant. It is only CT invariant, i.e., invariant by the transformation:

$$\begin{aligned} CT\colon x_4\to -x_4, \quad & x_k\to x_k, \quad B_4\to B_4, \quad B_k\to -B_k, \\ & g\to g, \quad \psi\to\gamma_3\gamma_1\psi^* \end{aligned} \qquad (108)$$

(CT is the so-called "weak" time reversal).

Of course, just like in the linear case, *the sign of g remains unchanged* under the CT transformation (108).

Now, introducing the change of variables (29) we get the two component spinor equations:

$$\begin{aligned} &\frac{1}{c}\frac{\partial\xi}{\partial t}-\mathbf{s}\cdot\nabla\xi-i\frac{g}{\hbar c}(W+\mathbf{s}\cdot\mathbf{B})\xi+i\frac{m(|\xi^+\eta|)c}{\hbar}(\eta^+\xi)\eta=0 \\ &\frac{1}{c}\frac{\partial\eta}{\partial t}+\mathbf{s}\cdot\nabla\eta+i\frac{g}{\hbar c}(W-\mathbf{s}\cdot\mathbf{B})\eta+i\frac{m(|\xi^+\eta|)c}{\hbar}(\xi^+\eta)\xi=0 \end{aligned} \qquad (109)$$

Making use of (33), (34), (35), and (36) we can verify once more that the system is P-invariant and gauge invariant, but not T and C invariant. The CT invariance is easily verified and the corresponding transformation is:

$$t\to -t, \qquad \mathbf{x}\to\mathbf{x}, \qquad W,\mathbf{B}\to -\mathbf{B}, \qquad g\to g,\ -is_2\xi^*\to\eta, \qquad is_2\eta^*\to\xi \qquad (110)$$

The most important property which appears immediately on the system (109) is that the left and right monopoles are no longer independent, but are *coupled* by the nonlinear term. Nevertheless, this nonlinear coupling is not so strong as it seems to be. One can verify indeed that not only the global magnetic current $i\bar{\psi}\gamma_\mu\gamma_5\psi$ is conserved (which is a consequence of the chiral gauge conservation) but moreover the relations (37) still holds, which means that both chiral isotropic currents are separately conserved. Besides, it turns out that not all pairs of (ξ,η) solutions are nonlinearly coupled and the exceptions are interesting.

The equations (107) or (109) show that the nonlinear coupling can disappear when both invariants Ω_1 and Ω_2 cancel simultaneously, which simply means, on (109), that:

$$\xi^+\eta = 0 \tag{111}$$

This happens in only three cases;

$$\xi = 0, \qquad \eta = 0, \qquad \xi = f(\mathbf{x}, t)s_2\eta^* \tag{112}$$

The first two cases correspond to a monopole alone, with a definite helicity which obeys one of the linear equations (31). But the most interesting case in (112) is the third one. In principle the factor $f(\mathbf{x}, t)$ is an arbitrary scalar function. But given that the equations for ξ and η are now separate and linear, we may postulate that ξ and η are both (and separately) normalized. Hence:

$$f(\mathbf{x}, t) = e^{i\phi(\mathbf{x},t)} \tag{113}$$

So, the factor f is no more than a phase factor which may be absorbed in the gauge of the external field and we can write, without loss of generality:

$$\xi = is_2\eta^* \tag{114}$$

But this is the charge conjugation relation (35). Therefore, the third case of cancellation of the nonlinear term in the equation (109) occurs for a pair monopole-antimonopole (i.e., monopoles with equal charges but opposite helicities).

Now, if we introduce (114) in (38), we find that the chiral currents are equal:

$$X_\mu = Y_\mu \tag{115}$$

From this equality and (39), it follows that:

$$J_\mu = 2X_\mu, \qquad \Sigma_\mu = 0 \tag{116}$$

which means that the electric current is isotropic, while the *magnetic current disappears.* Therefore, there will be no more observable current because our monopole is not supposed to be electrically charged and it seems that such a pair would be very difficult to observe. Therefore, it seems possible that we could be immersed in an *aether* made of such pairs of monopoles without having ever observed it. Such an aether would be, of course, quite different from a vacuum constituted by pairs with opposite charges. But even if the plausibility of this monopole aether were theoretically confirmed, the question would remain: how could we break its homogeneity and observe it?

10. PLANE WAVES AND DISPERSION RELATIONS

One can see that in (109), the phases of ξ and η are independent. We can thus consider the following plane waves (where a and b are constant spinors):

$$\xi = ae^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}, \qquad \eta = be^{i(\omega' t - \mathbf{k}'\cdot\mathbf{r})} \tag{117}$$

Introducing these expressions in (109) without external field, we find:

$$\begin{aligned} \left(\frac{\omega}{c} + \mathbf{s}\cdot\mathbf{k}\right) a + \frac{m(|a^+b|)c}{\hbar}(b^+a)b = 0 \\ \left(\frac{\omega'}{c} - \mathbf{s}\cdot\mathbf{k}'\right) b + \frac{m(|a^+b|)c}{\hbar}(a^+b)a = 0 \end{aligned} \tag{118}$$

If we multiply the first equation by $(\omega'/c - \mathbf{s}\cdot\mathbf{k}')$, introducing the relations:

$$\begin{aligned} \left(\frac{\omega'}{c} - \mathbf{s}\cdot\mathbf{k}'\right)\left(\frac{\omega}{c} + \mathbf{s}\cdot\mathbf{k}\right) = \Omega + \mathbf{s}\cdot\mathbf{K} \\ \Omega = \frac{\omega\omega'}{c^2} - \mathbf{k}\cdot\mathbf{k}', \qquad \mathbf{K} = \frac{1}{c}(\omega'\mathbf{k} - \omega\mathbf{k}') + i\mathbf{k}\times\mathbf{k}' \end{aligned} \tag{119}$$

we find:

$$(\Omega + \mathbf{s}\cdot\mathbf{K})a + \frac{m(|a^+b|)c}{\hbar}(b^+a)\left(\frac{\omega'}{c} - \mathbf{s}\cdot\mathbf{k}'\right) b = 0 \tag{120}$$

Then, using the second equation (118), it follows:

$$\left(\Omega + \mathbf{s}\cdot\mathbf{K} - \frac{M^2c^2}{\hbar^2}\right) a = 0 \tag{121}$$

with

$$M = m(|a^+b|)\times|a^+b| \tag{122}$$

In order to have nontrivial solutions in the equation (121), we must impose:

$$\det\left(\Omega + \mathbf{s}\cdot\mathbf{K} - \frac{M^2c^2}{\hbar^2}\right) = 0 \tag{123}$$

This is the dispersion relation, but it needs to be put in a more explicit form. Let us denote by K_1, K_2, K_3 the components of $\mathbf{K}$ and (123) becomes:

$$\det\begin{pmatrix} \Omega-\frac{M^2c^2}{\hbar^2}+K_3 & K_1-iK_2 \\ K_1+iK_2 & \Omega-\frac{M^2c^2}{\hbar^2}-K_3 \end{pmatrix}=0 \tag{124}$$

Hence we have:

$$\left(\Omega-\frac{M^2c^2}{\hbar^2}\right)^2-K^2=0 \tag{125}$$

But taking account of (119), we have:

$$K^2=\frac{1}{c^2}(\omega'\mathbf{k}-\omega\mathbf{k}')^2-(\mathbf{k}\times\mathbf{k}')^2 \tag{126}$$

which proves that K^2 is real, and it is easy to find that:

$$\Omega^2-K^2=\left(\frac{\omega^2}{c^2}-k^2\right)\left(\frac{\omega'^2}{c^2}-k'^2\right) \tag{127}$$

Consequently, the dispersion relation (125) takes the explicit form:

$$\left(\frac{\omega^2}{c^2}-k^2\right)\left(\frac{\omega_2'}{c^2}-k'^2\right)-2\left(\frac{\omega\omega'}{c^2}-\mathbf{k}\cdot\mathbf{k}'\right)\frac{M^2c^2}{\hbar^2}+\frac{M^4c^4}{\hbar^4}=0 \tag{128}$$

Two particular cases are especially interesting:

(1) $\omega=\omega'$, $k=k'$: both monopoles ξ and η have the *same phase.* The dispersion relation immediately reduces to:

$$\frac{\omega^2}{c^2}=k^2+\frac{M^2c^2}{\hbar^2} \tag{129}$$

This is the ordinary dispersion relation of a massive particle in quantum mechanics with a *proper mass M.* But in our case, this proper mass depends in general on the amplitudes a and b through the relation (122), except in the case when the function m in the equations (109) is:

$$m(|\xi^+\eta|)=\frac{m_0}{|\xi^+\eta|} \qquad (m_0=\text{Const}) \tag{130}$$

For this particular choice of m, M^2 reduces to the constant m_0^2 in the relation (129); it is worth noting that the equations (119) are then homogeneous in ξ and η, which means that ξ and η (respective, a and b) are only defined up to a normalization constant.

(2) $\omega = -\omega'$, $\mathbf{k} = -\mathbf{k}'$: the monopoles have opposite phases. Now the dispersion relation reduces to:

$$\frac{\omega^2}{c^2} = k^2 - \frac{M^2c^2}{\hbar^2} \tag{131}$$

This case corresponds to a supraluminal particle: a *tachyon*. In the terminology adopted in tachyon theory (Recami, 1979) we can say that the dispersion relation (129) corresponds to a *bradyon* and that the bradyon case and the tachyon case are linked by the limit case of a *luxon* which corresponds to $M = 0$, i.e.,

$$\xi^+\eta = a^+b = 0 \tag{132}$$

11. THE RELATIONS BETWEEN A CHIRAL GAUGE AND A TWISTED SPACE

In the absence of the special electromagnetic coupling which characterizes the magnetic monopole, the nonlinear equation (107) was already known under different forms. Let us particularize the Lagrangian (106), putting:

$$B_\mu = 0, \qquad \frac{1}{4}\frac{\mathcal{M}(\rho^2)c}{\hbar} = \frac{\lambda}{2}\rho^2 = \frac{\lambda}{2}(\Omega_1^2 + \Omega_2^2) \tag{133}$$

where λ is a constant. This nonlinear term may be written in two other forms using (21), so that we get for (107), omitting the electromagnetic field, the following three equivalent equations:

$$\gamma_\mu\, \partial_\mu\psi + \lambda[\bar\psi\psi - (\bar\psi\gamma_5\psi)\gamma_5]\psi = 0 \tag{134}$$

$$\gamma_\mu\, \partial_\mu\psi + \lambda(\bar\psi\gamma_\mu\psi)\gamma_\mu\psi = 0 \tag{135}$$

$$\gamma_\mu\, \partial_\mu\psi + \lambda(\bar\psi\gamma_5\gamma_\mu\psi)\gamma_5\gamma_\mu\psi = 0 \tag{136}$$

The equations (135) and (136) were studied by several authors (Finkelstein, 1951, 1956; Rañada, 1978), with an additional linear term of mass; the restriction of (135) to a $1+1$ dimension space is the well-known Thirring (Thirring, 1958) model (Rañada, 1984). The equation (136), without linear term, was considered by Heisenberg and coworkers as a basis of a theory of elementary particles (Heisenberg, 1958; Heisenberg et al., 1959). It is interesting to note that Heisenberg had previously considered the equation (Heisenberg, 1954; Heisenberg et al., 1955):

$$\gamma_\mu\, \partial_\mu\psi + \lambda(\psi^+\psi)\psi = 0 \tag{137}$$

which was studied more recently by Soler, Rañada, and Alvarez (Soler, 1970; Rañada and Soler, 1972; Alvarez and Soler, 1983) (with a linear term). But the great difference is that (137) is not invariant under the chiral gauge (10). We must recall that if one adds a linear term of mass to the equations (134), (135), or (136) they lose the chiral invariance and are no longer compatible with the monopole electromagnetic coupling (107).

The most interesting works, from the point of view of the present paper are the works of Weyl (Weyl, 1950), Kibble (Kibble, 1961), and Rodichev (Rodichev, 1961), who found the equation (136) from a geometric point of view.

The problem of Weyl was to compare the Dirac theory written in the ordinary metric relativistic theory (in which the connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ are expressed in terms of $g_{\mu\nu}$) and in the "mixed" theory in which the $\Gamma^{\lambda}_{\mu\nu}$ and the $g_{\mu\nu}$ are considered as independent quantities: he found that Dirac's equation retains its form in both frameworks *only if* one adds a nonlinear term which tends to $(\bar{\psi}\gamma_{\mu}\gamma_5\psi)\gamma_{\mu}\gamma_5\psi$ for a vanishing gravitational field. This result was later confirmed by Kibble, whose problem was to deduce the Einstein gravitational theory from the invariance under the inhomogeneous Lorentz group. It must be noted that, in a 5-dimensional theory, Rañada and Soler (Rañada and Soler, 1972) did not found the Weyl term but a term which looks like the one of (137) (expressed in a 5-dimensional space).

The interpretation of (136) by Rodichev is the following. He considers a *non symmetric* affine connection, i.e.,

$$S^{\lambda}_{[\mu\nu]} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \tag{138}$$

but with *rectilinear geodesics.* It is known that, contrary to the connection $\Gamma^{\lambda}_{\mu\nu}$ itself its antisymmetric part $S^{\lambda}_{[\mu\nu]}$ (when it exists) is a *tensor* and represents a *torsion* of the space. This signifies that, when $S^{\lambda}_{[\mu\nu]} \neq 0$ the image of a closed curve of an affine space in such a *twisted space* is always broken (i.e., a nonclosed curve) and that, if we try to squeeze the initial closed curve to infinitesimal dimensions, the magnitude of the *gap* between the ends of the image remains of the *second-order* with regard to the linear dimensions of the initial curve. In other words, the image of a loop is an arc of helicoide with a "thread" which remains of the same order as the *area* of the loop. Here lies the difference with a symmetric connection space, where the magnitude of a similar gap (or thread) would be of a higher infinitesimal order. These characteristics of a *twisted space* are in close analogy with those of a *spin fluid*: in the same manner, a *spin density* is not to be confused with a proper angular momentum due to the curl of an ordinary fluid motion, because the spin of a fluid droplet *dv* is of the same order as *dv* itself, while the proper angular momentum of the same droplet (which is the only one which exists in an ordinary fluid) is of the fifth-order of magnitude with respect to the linear dimensions of the droplet

(Halbwachs, 1960). The analogy between twisted spaces and spin fluids were guessed a long time ago, but now it has become more than a simple analogy, since the recent works of Ray and Smalley (Ray and Smalley, 1982, 1983; Smalley and Ray, 1984) who demonstrated it as a consequence of their geometrization of spin.

Let us go back to the idea of Rodichev (with some slight modifications). In such a flat twisted space, in cartesian coordinates, $\Gamma^{\lambda}_{\mu\nu}$ may be restricted to a completely antisymmetric tensor:

$$\Gamma_{\lambda\mu\nu} = \phi_{[\mu\nu\lambda]} \tag{138a}$$

and the same holds for $S_{[\lambda\mu\nu]}$ which is twice $\Gamma_{\lambda\mu\nu}$. Then, Rodichev introduces the following covariant derivative for the Dirac spinor:

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{i}{4} \phi_{\mu\nu\lambda} \gamma_\nu \gamma_\lambda \psi \tag{139}$$

and the Lagrangian density:

$$L = \tfrac{1}{2}\{\bar{\psi}\gamma_\mu \nabla_\mu \psi - (\nabla_\mu \bar{\psi})\gamma_\mu \psi\} \tag{140}$$

Already at this stage, we can make the following remark. L may be written, owing to (139), in the form:

$$L = \tfrac{1}{2}\left\{ \bar{\psi}\gamma_\mu \,\partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma_\mu \psi - \frac{i}{2} \phi_{\mu\nu\lambda} \bar{\psi}\gamma_\mu \gamma_\nu \gamma_\lambda \psi \right\} \tag{141}$$

Let us introduce the dual Φ_μ of $\phi_{\mu\nu\lambda}$, which is an axial vector:

$$\Phi_\mu = \frac{i}{3!} \varepsilon_{\mu\nu\lambda\sigma} \phi_{\nu\lambda\sigma} \tag{142}$$

Hence, L becomes:

$$L = \tfrac{1}{2}\{\bar{\psi}\gamma_\mu \,\partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma_\mu \psi - \tfrac{1}{2}\Phi_\mu \bar{\psi}\gamma_\mu \gamma_5 \psi\} \tag{143}$$

and the corresponding field equation is:

$$\gamma_\mu(\partial_\mu - \tfrac{1}{2}\Phi_\mu \gamma_5)\psi = 0 \tag{144}$$

This is exactly the equation (19) with:

$$\Phi_\mu = \frac{2g}{\hbar c} B_\mu \tag{145}$$

We may interpret this identification by the assertion that: when a monopole is plunged in an external electromagnetic field, the surrounding space becomes twisted for it, with an antisymmetric connection which is equal (apart from a constant factor) to the dual of the axial potential of the field.

But this was not the reasoning of Rodichev, whose work was not concerned with the monopole problem. He considered an Einstein variational principle with the following action integral:

$$S=\int (L-bR)\, d^4x \tag{146}$$

where L is the density (140) or (143); b is a constant and R the total curvature. The Riemann-Christoffel curvature tensor:

$$-R^{\mu}_{\nu\sigma\lambda}=\partial_\sigma\Gamma^{\mu}_{\nu\lambda}-\partial_\lambda\Gamma^{\mu}_{\nu\sigma}+\Gamma^{\mu}_{\rho\sigma}\Gamma^{\rho}_{\nu\lambda}-\Gamma^{\mu}_{\rho\lambda}\Gamma^{\rho}_{\nu\sigma} \tag{147}$$

gives immediately, in virtue of (138), (142):

$$R=\phi_{\lambda\mu\nu}\phi_{\lambda\mu\nu}=-6\Phi_\mu\Phi_\mu \tag{148}$$

Hence, the action integral has the expression:

$$S=\int \{\tfrac{1}{2}[\bar{\psi}\gamma_\mu\, \partial_\mu\psi-(\partial_\mu\bar{\psi})\gamma_\mu\psi-\tfrac{1}{2}\phi_\mu\bar{\psi}\gamma_\mu\gamma_5\psi]+6b\Phi_\mu\Phi_\mu\}d^4x \tag{149}$$

The variation with respect to ψ gives the equation (144) again and variating with respect to Φ_μ, one gets:

$$\Phi_\mu=\frac{1}{24b}\,\bar{\psi}\gamma_\mu\gamma_5\psi \tag{150}$$

and the equation (144) takes the form:

$$\gamma_\mu\,\partial_\mu\psi-\frac{1}{48b}\,(\bar{\psi}\gamma_\mu\gamma_5\psi)\gamma_\mu\gamma_5\psi=0 \tag{151}$$

We recognize the equation (136) with $\lambda=-1/48b$, which is equivalent to (134), i.e., equivalent to our equation (107) without external field and with the particular expression (133) for $\mathscr{M}(\rho^2)$. In consequence of (21), (22), (148), and (150), we have:

$$R=-\frac{1}{4b}\,(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\mu\gamma_5\psi)=\frac{1}{4b}\,(\Omega_1^2+\Omega_2^2) \tag{152}$$

In conclusion, we see that the nonlinear term we have introduced in the equation (107) on the basis of the chiral gauge invariance may be interpreted as the effect of a torsion of the space due to a self action of the monopole on itself. In such an interpretation, our fundamental chiral invariant $\Omega_1^2+\Omega_2^2$ is nothing but the total curvature of the twisted space (apart from a constant factor).

12. SOME CONCLUSIONS AND EXPERIMENTAL SUGGESTIONS

We have already noticed the obvious fact that the equations (19) and (31) of our monopole admit, as a particular case, the neutrino equations. But this particular case may be considered in two different ways. The first one is trivial: one can say that the equations of the monopole reduce to the one of a neutrino in the absence of electromagnetic field, which is not very interesting. The second way is a little adventurous and would require more careful thought but it seems to be worthy of note even at the present stage. We know, from the Dirac relation (66), that the magnetic charge g which appears in (19) and (31) is quantized. It is a multiple of elementary charge g_0:

$$g = ng_0, \qquad g_0 = \frac{\hbar c}{2e} \tag{153}$$

where e is the charge of the electron.

Therefore, our equations actually describe a family of monopoles, each of which corresponding to a value of n. But we suggest the hypothesis that they are in fact a kind of *excited states of only one particle, the neutrino*, so that the latter will correspond to a kind of ground state $n = 0$. One can ask: how could magnetism be created from nothing but an excitation? We have no conclusive answer to this question, but here is an argument. We may observe indeed that contrary to the electric charge which is always conserved in the quantum equations (in virtue of the phase invariance), this is not the case for the magnetic charge. We have seen that the chiral gauge invariance (14) which entails the conservation of the magnetic current (20) is only true in the *massless* Dirac equation. If we consider the Dirac equation of the electron with a mass m_0:

$$\gamma_\mu \left(\partial_\mu - i\frac{e}{\hbar c} A_\mu \right) \psi - \frac{m_0 c}{\hbar} \psi = 0 \tag{154}$$

we will not find a conservation law for the axial vector Σ_μ (22), but the Uhlenbeck and Laporte relation (Takabayasi, 1957):

$$\partial_\mu \Sigma_\mu + 2\frac{m_0 c}{\hbar} \Omega_2 = 0 \tag{155}$$

where Ω_2 is the Dirac pseudo-invariant (22). And if we rewrite (154) in the spinorial form:

$$\begin{aligned} (\pi_0 + \boldsymbol{\pi} \cdot \mathbf{s})\xi + m_0 c\eta &= 0 \\ (\pi_0 - \boldsymbol{\pi} \cdot \mathbf{s})\eta + m_0 c\xi &= 0 \end{aligned} \tag{156}$$

$$\pi_0 = \frac{1}{c}\left(i\hbar\frac{\partial}{\partial t} + e\phi\right), \qquad \boldsymbol{\pi} = -i\hbar\nabla + \frac{e}{c}\mathbf{A} \tag{157}$$

the Uhlenbeck and Laporte relation splits into the following two new relations, in terms of *chiral currents*:

$$\begin{aligned} &\frac{1}{c}\frac{\partial}{\partial t}(\xi^+\xi) - \nabla(\xi^+\mathbf{s}\xi) = \frac{m_0 c}{\hbar}\Omega_2 \\ &\frac{1}{c}\frac{\partial}{\partial t}(\eta^+\eta) + \nabla(\eta^+\mathbf{s}\eta) = -\frac{m_0 c}{\hbar}\Omega_2, \qquad \Omega_2 = i(\xi^+\eta - \eta^+\xi) \end{aligned} \tag{158}$$

These relations, by a substraction, give (155) again. As far as we have interpreted the axial and the chiral currents as magnetic currents, the relations (155) and (158) mean that there are sources of magnetism in the Dirac equation while, on the contrary, there are not any sources of electricity, because the electric current is always conservative. Therefore, it seems conceivable to consider a possible creation of magnetic poles in some particle reactions.

Then, if we admit the hypothesis that our monopoles are magnetically excited states of a neutrino, it is natural to imagine that they take part not only in electromagnetic, but also in weak interactions. Therefore, a new question arises: is it possible that some weak reactions produce such monopoles instead of neutrinos? Of course if it is so, there must be at least two classes of monopoles, corresponding to the excited states of ν_e and of ν_μ.

In the present paper we will not try to go further into these problems. Nevertheless, we may still add a remark.

Let us suppose, for example, that in the sun, a certain part of the fusion reactions:

$$p + p \rightarrow D + e^+ + \nu_e \tag{159}$$

produces monopoles, instead of neutrinos. Then, two possible consequences follow immediately: (a) Such processes could play a role in the magnetical aspects of the solar activity, in particular in the *sunspots*, in which could emerge strong monopole flows generated in the depths of the sun. (b) It seems natural to admit that these monopoles undergo an important energy loss in passing through matter.[2] Thus, contrary to the neutrinos, they either cannot reach the earth or if they make their way down to us, they probably have a very low energy, which could be under the reaction threshold of the

[2]Unfortunately even this simple assertion is only a conjecture because all the works devoted to such an energy loss are always concerned with heavy and slow monopoles, so that our case (massless monopoles) still remains unexplored.

devices constructed for the measurement of the solar neutrino flow. But it is a well-known fact that there is a still unexplained lack of registered solar neutrinos in comparison with the estimation of the flow emitted by the sun. A possible explanation of this lack is that only the "ground state" neutrinos produced by the reaction (159) were observed until now, and not their excited states: the massless monopoles.

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